Section 2.2 Operations on Sets

Purpose of Section: To introduce the set operations of union (\( \cup \)), intersection (\( \cap \)), difference (\( \setminus \)) and relative complement (\( ^c \)). We prove a number of important laws of sets, such as the distribution of "\( \cup \)" over "\( \cap \)", and vice versa, as well as De Morgan’s Laws.

Union, Intersection and Complement

In traditional arithmetic and algebra, we carry out the binary operations of + and \( \times \) on numbers. In logic, we have the analogous binary operations of \( \lor \) and \( \land \) on sentences. In set theory we have the binary operations of union \( \cup \) and intersection \( \cap \) of sets, which in a sense are analogous to the ones in arithmetic and sentential logic.

**Definition (Union) 2.1:** The union of two sets A and B, denoted \( A \cup B \), is the set of elements that belong to A or B or both. Symbolically

\[
A \cup B = \{ x \mid x \in A \lor x \in B \}
\]

**Definition (Intersection) 2.2:** The intersection of two sets A and B, denoted \( A \cap B \), is the set of elements that belong to A and B. Symbolically

\[
A \cap B = \{ x \mid x \in A \land x \in B \}
\]
Normally, in discussions involving sets we imagine all sets under consideration as belonging to some universal set, like the real number, complex numbers, and so on, which we denote by \( U \).

**Definition (Complement) 2.3:** The compliment of \( A \), denoted \( A^c \) is the set of elements belonging to the universal set \( U \) but not \( A \).

Symbolically

\[
A^c = \{ x \mid x \in U \land x \notin A \}. 
\]

**Definition (Relative Complement or Difference) 2.4:** The relative complement of \( A \) in \( B \), denoted, \( B \setminus A \), is the set of elements in \( B \) but not in \( A \).

Symbolically

\[
B \setminus A = \{ x \mid x \in B \land x \notin A \}. 
\]

**Venn Diagrams**

The concepts of union, intersection and relative complement of sets are illustrated graphically by use of **Venn** diagrams. Each Venn diagram begins with an oval representing the universal set, a set that contains all elements of
discussion, maybe the real number, complex numbers, and so on. Then, each set in the discussion is represented by a circle, where elements belonging to more than one set are placed in sections where circles overlap.

**Margin Notes:** Venn diagrams were the invention of British logician John Venn (1834-1923) who made major contributions to logic and probability. John Venn was an ordained minister but gave up the ministry in 1883 to concentrate on mathematics and logic.

Figure 1 illustrates typical Venn diagrams for two overlapping sets.

![Venn Diagrams for Two Sets](image)

Figure 2 illustrates Venn diagrams for three sets.
The properties of " $\cup$ " and " $\cap$ " in set theory have their counterparts in the properties of " $\lor$ " and " $\land$ " in sentential logic.

The following illustrates these counterparts.

<table>
<thead>
<tr>
<th>Tautology</th>
<th>Set Equivalence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \lor Q \Leftrightarrow Q \lor P$</td>
<td>$P \cup Q = Q \cup P$</td>
</tr>
<tr>
<td>$P \land Q \Leftrightarrow Q \land P$</td>
<td>$P \cap Q = Q \cap P$</td>
</tr>
<tr>
<td>$P \lor (Q \lor R) \Leftrightarrow (P \lor Q) \lor R$</td>
<td>$P \cup (Q \cup R) = (P \cup Q) \cup R$</td>
</tr>
<tr>
<td>$P \land (Q \land R) \Leftrightarrow (P \land Q) \land R$</td>
<td>$P \cap (Q \cap R) = (P \cap Q) \cap R$</td>
</tr>
<tr>
<td>$P \lor (Q \lor R) \leftrightarrow (P \lor Q) \lor (P \lor R)$</td>
<td>$P \cup (Q \cup R) = (P \cup Q) \cup (P \lor R)$</td>
</tr>
<tr>
<td>$P \land (Q \land R) \leftrightarrow (P \land Q) \land (P \land R)$</td>
<td>$P \cap (Q \cap R) = (P \cap Q) \cap (P \cap R)$</td>
</tr>
<tr>
<td>$P \lor P \leftrightarrow P$</td>
<td>$P \cup P = P$</td>
</tr>
<tr>
<td>$P \land P \leftrightarrow P$</td>
<td>$P \cap P = P$</td>
</tr>
<tr>
<td>$\neg (P \land Q) \leftrightarrow \neg P \lor \neg Q$</td>
<td>$(P \land Q)^c = P^c \lor Q^c$</td>
</tr>
<tr>
<td>$\neg (P \lor Q) \leftrightarrow \neg P \land \neg Q$</td>
<td>$(P \lor Q)^c = P^c \land Q^c$</td>
</tr>
</tbody>
</table>
Naive versus Axiomatic Set Theory: Naive set theory studies elementary properties of sets, such as the union, intersection, Venn diagrams, De Morgan’s laws, partially ordered sets, and so on using general intuition and a minimal number of axioms. Unfortunately for a rigorous study of sets, naïve set theory is inconsistent: i.e. it is possible to prove contradictions, the most famous contradiction of paradox being the “barber paradox” posed by English logician Bertrand Russell. Axiomatic set theory was created to place set theory on a set of form axiomatic foundation where the axioms are consistent (no internal contradictions) and independent (no one axiom could be proven from the others). A common axiom system of logic used by logicians is the Zermelo-Fraenkel (ZF) axioms, named after logicians Ernst Zermelo (1871-1953) and Abraham Fraenkel (1891-1965).

Margin Note: The symbols ∪ and ∩ for set union and intersection is due to the Italian mathematician Giuseppe Peano (1858-1932).

Theorem 2.1: For A and B sets, we have A ⊆ B → B^c ⊆ A^c.

Proof: To show A ⊆ B → B^c ⊆ A^c we show B^c ⊆ A^c using the hypothesis A ⊆ B as assistance when needed. Letting x ∈ B^c says that x does not belong to B, but the hypothesis A ⊆ B tells us that A is contained in B and hence x does not belong to A as well, or in other words x ∉ A. Hence, we have B^c ⊆ A^c proved.

Theorem 2.2: Let A, B and C be sets. Then "∩" distributes over "∪". That is

A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

Proof: We show two inclusions:

a) (⊆): A ∩ (B ∪ C) ⊆ (A ∩ B) ∪ (A ∩ C)
b) (⊇): A ∩ (B ∪ C) ⊇ (A ∩ B) ∪ (A ∩ C)
To show a) we write
\[ x \in A \cap (B \cup C) \rightarrow (x \in A) \text{ and } (x \in B \cup C) \]
\[ \rightarrow (x \in A) \text{ and } (x \in B \text{ or } x \in C) \]
\[ \rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \]
The last line states that
\[ x \in (A \cap B) \cup (A \cap C) \]
which proves a).

To prove b) we argue backward.
Let \( x \in (A \cap B) \cup (A \cap C) \rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \)
\[ \rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \]
\[ \rightarrow (x \in A) \text{ and } (x \in B \text{ or } x \in C) \text{ (WHY ???)} \]
\[ \rightarrow (x \in A \text{ and } x \in B \cup C) \]
\[ \rightarrow x \in A \cap (B \cup C) \]
and so \((A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)\). Thus b) is verified and so we have proven the desired result.

**De Morgan’s Laws**

We saw in sentential logic the tautologies
\[ \neg (P \land Q) \equiv \neg P \lor \neg Q \text{ and } \neg (P \lor Q) \equiv \neg P \land \neg Q \]
The analogous identities for sets are called **De Morgan’s laws**.

**Theorem (De Morgan’s Law) 2.3:** Let A and B be sets then
\[ (P \cup Q)^c = P^c \cap Q^c \]
\[ (P \land Q)^c = P^c \cup Q^c \]

**Proof:** We prove the first De Morgan Law. We need to show the following two inclusions
\[ (P \cup Q)^c \subseteq P^c \cap Q^c \text{ and } (P \cup Q)^c \supseteq P^c \cap Q^c \].
We start with the first inclusion. Let
\[ x \in (P \cup Q)^c \rightarrow x \notin P \cup Q \]
\[ \rightarrow x \notin P \text{ and } x \notin Q \quad \text{(why ?)} \]
\[ \rightarrow x \in P^c \text{ and } x \in Q^c \]
\[ \rightarrow x \in (P^c \cap Q^c) \]
This proves \((P \cup Q)^c \subseteq P^c \cap Q^c\).

To show the second inclusion \((P \cup Q)^c \supseteq P^c \cap Q^c\), we let \(x\) be an arbitrary element from \(P^c \cap Q^c\) then the following chain of implications are true
\[ x \in (P^c \cap Q^c) \rightarrow x \in P^c \text{ and } x \in Q^c \]
\[ \rightarrow x \notin P \text{ and } x \notin Q \]
\[ \rightarrow x \notin P \cup Q \]
\[ \rightarrow x \in (P \cup Q)^c. \]

Hence, \((P \cup Q)^c \supseteq P^c \cap Q^c\). These two inclusions complete the verification of de Morgan's first Law. Arguing similarly you will be able to complete verification of second De Morgan's Law. END
Problems 2.2

1. Let $A$, $B$ and $C$ be arbitrary subsets of a universe $U$. Prove the following.
   a) $A \subseteq A$
   b) $A \cup \emptyset = A$
   c) $A \cap \emptyset = \emptyset$
   d) $\emptyset ^ c = U$
   e) $A \subseteq B$ iff $A ^ c \cup B = U$.
   f) $A \cap U = A$
   g) $A \cap A ^ c = \emptyset$
   h) $A \subseteq B \Rightarrow A \cup B = B$
   i) $A \cup A = A \cap A$
   j) $(A ^ c) ^ c = A$

2. (Find the Sets) Let the natural numbers $N$ be the universal set and
   $E = \{2, 4, 6, \ldots\}$, $O = \{1, 3, 5, \ldots\}$, $F = \{5, 10, 15, \ldots\}$, $P = \{2, 3, 5, 7, 11, \ldots\}$ (prime numbers). Find the following.
   a) $E \cap O$
   b) $E \cup F ^ c$
   c) $(E \cup F) ^ c$
   d) $(P \cap F ^ c) \cup E$
   e) $(E \cup O) \cap P$
   f) $(O \cap E ^ c) \cup P$

3. (Symmetric Difference) The symmetric difference to two sets $A$ and $B$ is the set of elements that belong to one of the sets but not both and is denoted by
   $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

Show the following. Draw Venn diagrams to illustrate your proof.
a) \( A \oplus B = B \oplus A \)

b) \( A \oplus (B \oplus C) = (A \oplus B) \oplus C \)

c) \( A \cup B = (A \oplus B) \oplus (A \cup B) \)

d) \( A \cap B = \emptyset \iff A \oplus B = A \cup B \)

4. (Difference Problems) Simplify the following.
   a) \( A - (B - C) = \)
   b) \( A - (B -(C \cup D)) = \)
   c) \( A - (B -(C - D)) = \)
   d) \( (A \cap (B \cup C))^c = \)

5. (NASC for Disjoint Sets) Prove that a necessary and sufficient condition for \( A \) and \( B \) to be disjoint is \( A \setminus B = A \).

6. Distributive Law: Prove that if \( A, B \) and \( C \) are sets, then "\( \cup \)" distributes over "\( \cap \)". That is
   \[
   A \cup (B \cap C) = (A \cup B) \cap (A \cup C).
   \]

7. (De Morgan's Law) Prove De Morgan's Law
   \[
   (A \cap B)^c = A^c \cup B^c.\]

8. (Sets and Arithmetic) Compare the set operations of union, intersection, and subtraction with the arithmetic operations of addition, multiplication and subtractions of numbers. In what ways are they similar? In what ways are they different?

9. (Sets and Sentential Logic) Compare the set operations of \( \cap, \cup, \setminus \) with the sentential logic operations of \( \land, \lor, \neg \). In what ways are they similar? In what ways are they different?
10. **(Computer Representation of Sets)** Finite sets can be represented efficiently by vectors of 0s and 1. For example, suppose we are interested in subsets of a finite universe $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$. We represent a subset like $A = \{2, 3, 6\}$ as a vector of 0s and 1s with 1's in the 2nd, 3rd, and 6th positions, and zeros elsewhere. That is $0110010$. In the following problems, if a set $A \subseteq U$ is provided, find the binary vector corresponding to the set; if a binary vector is given, find the corresponding subset of $U$.

a) $1001111$

b) $0000000$

c) $1111111$

d) $\{1,5,7\}$

e) $U$

f) $1000111 + 1100000$ (assume $1+1 = 1$)

g) $0000001 + 1110000$