Section 1.3: Rules of Logic and proofs in Propositional calculus

Purpose of Section: To introduce propositional forms and inference rules for simple propositional calculus. We will state some basic logic rules (inference rules) and see how it can be use to write some simple proofs.

Propositional forms

We have used letters $P$, $Q$, $R$ ... to represent propositions. We will call letters representing propositions *propositional variables*. We will assign to this variables propositions via the symbol" := " as in this example:

Example:

$$P := \text{"Pig can fly"}$$
$$R := \text{"10 + 7 > 12"}$$

According to sections 1.1 and 1.2 we know that $P \land R$ and $P \rightarrow R$ both are propositions even more we know that $P \land R$ is false proposition were as $P \rightarrow R$ is true. What about $R \land S$, can you tell whether it is true or false? As soon as we know what proposition is assigned to $S$ we will be able to claim truth value of $R \land S$ before that it will be *propositional form* or *propositional expression* according the following description.

To introduce *propositional expressions* or *propositional forms* we begin with *an alphabet*, which consists of

a) letters (we call them propositional variables or simply variables) $P$, $Q$, $R$, ... $x$, $y$, $z$

b) and some additional symbols $\land$, $\lor$, $\rightarrow$, $\leftrightarrow$, $\sim$ and pair of parentheses ( ).

Intuitively we characterize the *propositional expressions* as follows:

1. Any propositional variable is propositional expression.

2. If $x$ and $y$ are propositional expressions then

$$x \land y, x \lor y, \sim x, x \rightarrow y, x \leftrightarrow y$$

all are propositional expressions.

3. Any propositional expression is one of the forms 1. or 2.

Once again a propositional expression does not have a meaning until the propositional variables involved in the expression are replaced by propositions.
We will evaluate propositional expressions by completing the truth tables, like in 1.1 and 1.2, assuming that all propositional variables are replaced by propositions. To avoid possible ambiguity we will follow the following order of evaluations of operations rules

**Order of operations**

To interpret a propositional expression, we read it from left to right by using the following precedence rules:

1. propositional expressions within parentheses (innermost first)
2. negations,
3. conjunctions,
4. disjunctions,
5. conditionals,
6. biconditionals

**Example:** To interpret \( \neg P \lor Q \land R \) first we apply negation then conjunction and then followed disjunction. It also can be parenthesized as follows \((\neg P) \lor (Q \land R)\).

**Rules of Inference (Rules of Logic)**

In this section we are about to write some basic "proofs". For that we need rules of logic. The first one we call rule of inference or valid argument is most common way of expressing proofs in mathematics.

**Definition** Let \( H_1, H_2, \ldots, H_k \) and \( Q \) be propositional expressions. A propositional expression \( Q \) is said to be *logical consequence* of \( H_1, H_2, \ldots, H_k \) if \( Q \) is true whenever all \( H_1, H_2, \ldots, H_k \) are evaluated to be true.

This relationship is expressed symbolically by writing

\[ H_1, H_2, \ldots, H_k \vdash Q. \]

This is, what is called rule of inference or valid argument. It is normally read as

"\( H_1, H_2, \ldots, \) and \( H_k \) yield \( Q \) ".

**Example:**

a) \( P, P \to Q \vdash Q \)
b) \( P, Q \models P \land Q \)

c) \( P, Q \models P \lor Q \)

We will explore two methods of verifying valid arguments or rules of inference. The first is based on truth table. Let assume

\[ H_1, H_2, \ldots, H_k \models Q. \]

That is if all \( H_1, H_2, \ldots, H_k \) are true, then \( Q \) is true. That means

\[ H_1 \land H_2 \land \ldots \land H_k \rightarrow Q \]

is true and even more

\[ H_1 \land H_2 \land \ldots \land H_k \rightarrow Q \]

is tautology and vice versa.

**Example:** The following truth table shows \( P, P \rightarrow Q \models Q \) since it proves that \( (P \land (P \rightarrow Q)) \rightarrow Q \) is a tautology

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \rightarrow Q )</th>
<th>( P \land (P \rightarrow Q) )</th>
<th>( (P \land (P \rightarrow Q)) \rightarrow Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**Note:** If for some propositional forms \( H_1, H_2, \ldots, H_k \) and \( Q \) we have all \( H_1, H_2, \ldots, H_k \) are evaluated to be true and \( Q \) false, then we say that \( Q \) does not logically follow from \( H_1, H_2, \ldots, H_k \). We will use the following notation to indicate that by

\[ H_1, H_2, \ldots, H_k \models Q \]

(we call this invalid argument).

**Example:** Show that \( Q, P \rightarrow Q \models P \) (is not valid argument).

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \rightarrow Q )</th>
<th>( Q \land (P \rightarrow Q) )</th>
<th>( (Q \land (P \rightarrow Q)) \rightarrow P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
Theorem (RL): Let $P$, $Q$, $R$, and $S$ be propositional expressions then the followings are valid arguments (rules of logic)

1. Modus Ponens [MP]
   
   $P \rightarrow Q$, $P \models Q$

2. Modus Tolens [MT]

   $P \rightarrow Q$, $\neg Q \models \neg P$

3. Constructive Dilemma [CD]

   $(P \rightarrow Q) \land (R \rightarrow S)$, $P \lor R \models Q \lor S$

4. Simplification [S]

   $P \land Q \models P$

5. Addition [ADD]

   $P \models P \lor Q$

6. Conjunction [C]

   $P, Q \models P \land Q$

7. Disjunctive Syllogism [DS]

   $P \lor Q$, $\neg P \models Q$

8. Destructive Dilemma [DD]

   $(P \rightarrow Q) \land (R \rightarrow S)$, $\neg Q \lor \neg S \models \neg P \lor \neg R$

9. Transitivity [TR]

   $P \rightarrow Q$, $Q \rightarrow R \models P \rightarrow R$

Proof: Complete corresponding truth table for each rule of inference listed above.
Most of you remember proofs from your High School geometry class. Let me recall the structure of most of these proofs; it starts with the statements what is given (listing of hypotheses) and ends always with the desired outcome (conclusion of theorem). Between these two ends is filled with chain of statements, each following from earlier statements. We will generalize this as a second approach of verifying valid arguments.

**Definition.** A proof of $Q$ from $H_1$, $H_2$, ... $H_k$ is finite sequence of propositional forms $Q_1$, $Q_2$, ... $Q_n$ such that $Q_n$ is same as $Q$ and every $Q_j$ is either one of $H_i$, ($i = 1, 2, ..., k$) or it follows from the proceedings by the logic rules.

Following two examples shows how to complete the proof of the $H_1, H_2, ... H_k \vdash Q$.

**Note:** In these proofs we will follow the following formats: We begin with by listing all the hypotheses (marked as Hyp), then the sequence of propositional forms followed by the reason (short description of rules) that allowed that proposition to be included in proof in the same line and end with the conclusion. To make referencing easier we will number the lines and use abbreviated names of logic rules specified in the Theorem (RL1).

**Example:**

a) Prove $(P \lor Q) \rightarrow (Q \land R), P \vdash Q$.

Proof:

1. $(P \lor Q) \rightarrow (Q \land R)$  Hyp
2. $P$  Hyp
3. $P \lor Q$  3 Add
4. $Q \land R$  3, 1 MP
5. $Q$  4 S.

**END**
b) Prove \((P \rightarrow Q), P \lor (R \rightarrow S), \sim Q \vdash (R \rightarrow S)\)

Proof:
1. \((P \rightarrow Q)\) Hyp
2. \(P \lor (R \rightarrow S)\) Hyp
3. \(\sim Q\) Hyp
4. \(\sim P\) 1, 3 MT
5. \((R \rightarrow S)\) 2, 4 DS

Second Type of Logic rules (Rules of Replacement)

Recall that if two propositional expressions \(P\) and \(Q\) are logically equivalent \(P \equiv Q\) if they have same truth tables. Now suppose \(P \equiv Q\) and \(P\) appears in a propositional expression \(R\). If in \(R\) some of the appearances of \(P\) is replaced by \(Q\) then the new resulting propositional expression \(R'\) is logically equivalent to \(R\). to make proof writing more flexible we will extend rules of logic by adding some simple rules of replacement listed in the following theorem.

**Theorem (RR).** Let \(P\), \(Q\), and \(R\) be propositional expressions

1. **Commutative Law [Com]**
   \[
   P \land Q \equiv Q \land P \\
   P \lor Q \equiv Q \lor P
   \]

2. **Associative Law [Assoc]**
   \[
   (P \land Q) \land R \equiv P \land (Q \land R) \\
   (P \lor Q) \lor R \equiv P \lor (Q \lor R)
   \]

3. **Distributive Law [Dist]**
P ∧ (Q ∨ R) ≡ (P ∧ Q) ∨ (P ∧ R)
P ∨ (Q ∧ R) ≡ (P ∨ Q) ∧ (P ∨ R)

4. Contraapositive Law [Contr]

(P → Q) ≡ (¬ Q → ¬ P)

5. DeMorgan Law [DeM]

¬ (P ∧ Q) ≡ (¬ P ∨ ¬ Q)
¬ (P ∨ Q) ≡ (¬ P ∧ ¬ Q)

6. Double Negation [DN]

¬ ¬ (P) ≡ P

7. Implication Law [Impl]

(P → Q) ≡ (¬ P ∨ Q)

8. Equivalence Law [Equiv]

P ↔ Q ≡ (P → Q) ∧ (Q → P)
P ↔ Q ≡ (P ∧ Q) ∨ (¬ Q ∧ ¬ P)

9. Exportation [Exp]

(P ∧ Q) → R ≡ P → (Q → R)

10. Tautology [Taut]

P ∧ P ≡ P
P ∨ P ≡ P

The following example uses number of replacement rules in combination with inference rules.

Example: Prove (P → Q), (R → Q) ├ (P ∨ R) → Q
Proof:

1. \( P \rightarrow Q \) Hyp
2. \( R \rightarrow Q \) Hyp
3. \( \sim P \lor Q \) 1 Impl
4. \( \sim R \lor Q \) 2 Impl
5. \( (\sim P \lor Q) \land (\sim R \lor Q) \) 3,4 Conj
6. \( (Q \lor \sim P) \land (Q \lor \sim R) \) 5 Com
7. \( Q \lor (\sim P \land \sim R) \) 6 Dist
8. \( Q \lor \sim (P \land R) \) 7 DeM
9. \( \sim (P \land R) \lor Q \) 8 Com
10. \( (P \lor R) \rightarrow Q \) 9 Impl

Problems:

1. Use truth tables to prove the rules of inference listed in Theorem (RL).

2. Use truth table to show that the followings are valid arguments.
   a) \( P \rightarrow Q, P \models Q \lor R \)
   b) \( P \rightarrow Q, Q \rightarrow R, P \models R \)
   c) \( P, \sim (P \land Q) \models \sim Q \)
   d) \( (P \lor Q) \rightarrow R, P \models R \land S \)

3. Show that the following is true by specifying a single line of a truth table
   a) \( (P \land Q) \rightarrow R, P \models ? R \)
   b) \( (Q \lor R) \rightarrow P, P \models ? Q \)
   c) \( (P \land Q) \rightarrow R \models ? Q \rightarrow R \)
   d) \( P \lor R, P \leftrightarrow S \models ? R \land S \)
4. Identify the inference rule to justify the following:

a) \( P, (Q \lor R) \vdash P \land (Q \lor R) \)

b) \( (P \to Q) \to R, R \to S, \vdash (P \to Q) \to S \)

c) \( P \to \sim Q, \sim \sim Q \vdash \sim P \)

d) \( (P \to Q) \land (Q \to R), \sim Q \lor \sim R \vdash \sim P \lor \sim Q \)

5. Find proofs for the following

a) \( (P \to Q), (Q \to R), \sim R \vdash \sim P \)

b) \( (P \to Q), (R \to S), \sim Q \lor \sim S \vdash \sim P \lor \sim R \)

c) \( (P \to Q), (Q \to R), (R \to S), (S \to T), P \lor R, \sim R \vdash T \)

d) \( (P \to Q), (P \lor (R \to S)), \sim Q \vdash (R \land (Q \lor \sim P)) \)

6. Use truth tables to prove the rules of inference listed in Theorem (RR).

7. Prove the following by using truth tables:

a) \( (P \to Q) \to (S \to R) \equiv ((P \to Q) \land S) \to R \)

b) \( (P \land Q) \to R \equiv (P \land \sim R) \to \sim Q \)

c) \( P \land Q \equiv (P \leftrightarrow Q) \land (P \lor Q) \)

d) \( (P \to Q) \land R \equiv R \land (Q \lor \sim P) \)

8. Proof each of the following:

a) \( P \vdash \sim Q \to P \)

b) \( P \to (Q \land R) \vdash P \to Q \)

c) \( (P \lor Q) \to R \vdash \sim R \to \sim Q \)

d) \( P \leftrightarrow (Q \land R) \vdash P \to Q \)

e) \( P \to Q, P \to R, \vdash P \to (Q \land R) \)

f) \( P \leftrightarrow Q, \sim P \vdash \sim Q \)