Section 1.5 Mathematical Proofs

Purpose of Section: Most theorems in mathematics take the form of an implication \( P \rightarrow Q \) or as a biconditional \( P \leftrightarrow Q \), where the biconditional can be verified by proving both \( P \rightarrow Q \) and \( P \leftrightarrow Q \). We will study a variety of ways of proving \( P \rightarrow Q \) including a direct proof, proof by contrapositive, and three variations of proof by contradiction, including proof by reductio ad absurdum.

Introduction

A mathematical proof is an argument that convinces someone that something is true. Exactly what constitutes a mathematical proof is not static. More precisely, it is a sentence that can be demonstrated to be true by accepted logical operations.

A theorem\(^1\) is a true sentence which shows the relationship between mathematical ideas, and a justification that the theorem is true is called a proof of the theorem. The validity of a theorem can be traced back to a collection of ideas and concepts which considered so self-evident that their truth value is taken for granted. Such accepted truths are called axioms. Every branch of mathematics be it real or complex analysis, algebra, geometry, or applied mathematics is based on a collection of axioms, some the underlying axioms of logic, and others specific to the given area of investigation.

Types of Proofs

Many theorems in mathematics have the form of an implication \( P \rightarrow Q \), where one assumes the validity of \( P \), then with the aid of existing mathematical facts and theorems as well as laws of logic and reasoning, arrive at the conclusion \( Q \). Although the goal is always to “go from \( P \) to \( Q \),” is more than one valid way of

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\(^1\) The Hungarian mathematician Paul Erdos (1913-1996) once said that a mathematician is a machine for converting coffee into theorems.
achieving this goal. Five equivalent ways to proving the implication $P \rightarrow Q$ are shown in Table 1

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<thead>
<tr>
<th>Five Equivalent Forms of Implication</th>
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<tr>
<td>$P \implies Q$</td>
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<td>$\sim Q \implies \sim P$</td>
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<td>$(P \land \sim Q) \implies Q$</td>
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<td>$(P \land \sim Q) \implies \sim P$</td>
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<td>$(P \land \sim Q) \implies R \land \sim R$</td>
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Table 1

A truth table in Table 2 verifies the equivalence of these five forms of the basic implication.

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*same truth values*

Five Equivalent Sentences for Implication Table 1.

(Table 2)

In this and the next section we will demonstrate different methods of proof on a variety of theorems. We begin with a very simple proof.
Example 1: (Very Simple Proof) If \( p \) is an integer greater than 2, then \( p \) must be an odd integer.

Proof:
We show \( p \) is odd by showing it is not divisible by 2. Since \( p \) is prime it is only divisible by 1 and itself, and since \( p \) is not 2 it is not divisible by 2. END

Example 2: (Direct Proof) If \( n \) is an odd natural number, then \( n^2 \) is odd.

Proof:
An integer \( n \) is called odd if it is of the form \( n = 2k +1 \) for some integer \( k \), then since we assumed \( n \) odd, we can write \( n = 2k +1 \). Squaring gives
\[
    n^2 = (2k +1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
\]
Since \( k \) is a natural number we know \( s = 2k^2 + 2k \) is a natural number and so
\[
    n^2 = 2s +1
\]
which proves that \( 2n \) is odd. END

There are no magic bullets in proving theorems. Sometimes however the result to be proven provides the starting point and the theorem can be proven by working backwards. The idea is to work backwards until one arrives at an obviously true statement and then turn around and work to the end. The following theorem provides a good example of this technique.

Example 3 (Backwards Proof): Prove that for any two positive real numbers \( X \) and \( Y \), one has
\[
    \sqrt{XY} \leq \frac{X + Y}{2}
\]
Proof:

We begin by writing the conclusion in equivalent algebraic form

\[ X + Y \geq 2\sqrt{XY} \]

and using the fact that \( X, Y \) are positive we have another equivalent form

\[ X + Y - 2\sqrt{XY} \geq 0 \]

or by factoring

\[ (\sqrt{X} + \sqrt{Y})^2 \geq 0 \]

But this statement is obviously true and equivalent to the result we wish to prove. Hence we have proven the theorem. **END**

**Margin Note:** A conjecture is a mathematical statement which is believed to be true but has not been proven. Once proven it will be called a theorem. The Goldbach conjecture is one of the oldest unsolved problems in mathematics, which states that every even integer greater than 2 can be written as the sum of two (not necessarily distinct) primes. For example \( 4 = 2 + 2, \ 6 = 3 + 3, \ 8 = 3 + 5, \ 10 = 3 + 7 = 5 + 5, \ldots \) and so on.

**Analysis of Proof Techniques**

**Direct Proof** \([ H_1, H_2, \ldots H_k \vdash P \rightarrow Q ]\): A direct proof starts with the given assumption \( P \) and uses existing facts \( H_1, H_2, \ldots H_k \) to establish the truth of the conclusion \( Q \). More formally, let \( H_1, H_2, \ldots H_k, P \) and \( Q \) be a propositional expressions then

\[ H_1, H_2, \ldots H_k \vdash P \rightarrow Q \]

if and only if

\[ H_1, H_2, \ldots H_k, P \vdash Q . \]

The Rule of **Direct Proof** \([DP]\) (often called **Deduction Theorem**) can be shown to be true by use of **Exportation** (see the **Theorem RR** from section 1.4):

\[ H_1 \land H_2 \land \ldots \land H_k \rightarrow (P \rightarrow Q) \equiv (H_1 \land H_2 \land \ldots \land H_k \land P ) \rightarrow Q . \]
Let us demonstrate use if DP for formal two-column proofs:

Example 4: Prove \((P \lor Q) \rightarrow (R \land S) \vdash P \rightarrow R\).

Proof:

1. \((P \lor Q) \rightarrow (Q \land R)\) Hyp
   (Goal \(P \rightarrow R\))
2. \(P\) Assumption
3. \(P \lor Q\) 2 Add
4. \(Q \land R\) 2, 1 MP
5. \(R\) 4 S
6. \(P \rightarrow R\). 2-5 DP

END

The proof of \(P \rightarrow Q\) is a part of given proof. Blue part (lines 2 – 5 which begins with the assumption of \(P\) uses that assumption as a fact only in that part, (not elsewhere) which proves the conditional. At the point when we arrive at the conclusion \(R\) we have proven implication \(P \rightarrow R\). Direct Method is the preferred method for proving implications. Basically it says that to prove implication \(P \rightarrow Q\) given certain hypotheses, add \(P\) to the list of hypotheses and demonstrate \(Q\). In other words;

Assume the hypotheses ------------ Show the conclusion.

We can use this approach to write paragraphs proofs also. Always follow the following steps while writing paragraph proofs:

1. Identify hypothesis and conclusion of implication,
2. assume the hypothesis
3. translate the hypothesis (replace more useable equivalent form),
4. write comment regarding what must be shown,
5. translate the comment (replace more useable equivalent form)
6. deduce the conclusion
Example 5: Let \( x \) be an integer. Prove that if \( x \) is an odd integer then \( x + 1 \) is even integer.

Let \( P:="x \) is odd integer" \) (hypothesis) and
\( Q:"x + 1 \) is even integer \). (conclusion).

What we want
\[ \vdash P \to Q \]

Assume \( P \):

Translate: by definition \( x \) is odd if \( x = 2k + 1 \) for some integer \( k \)

Then
\[
\begin{align*}
x + 1 &= (2k + 1) + 1 \\
&= 2k + 2 \\
&= 2(k + 1)
\end{align*}
\]

Indirect proofs refer to proof by contrapositive or proof by contradiction which we introduce here. A contrapositive proof or proof by contrapositive for conditional proposition \( P \to Q \) one makes use of the tautology \( (P \to Q) \leftrightarrow (\sim Q \to \sim P) \). Since \( P \to Q \) and \( \sim Q \to \sim P \) are logically equivalent we first give a direct proof of \( \sim Q \to \sim P \) and then conclude that \( P \to Q \). One type of theorem where proof by contrapositive is often used is where the conclusion \( Q \) states something does not exist. Here it is convenient to assume the contrary; that the object in question does exist.

Here is the structure of contrapositive proof of \( P \to Q \)

Assume, \( \sim Q, \ldots \)

Therefore, \( \sim P \).

Thus, \( \sim Q \to \sim P \)

Therefore, \( P \to Q \).

Let \( H_1, H_2, \ldots H_k \) and \( Q \) be a propositional expressions then
\[
H_1, H_2, \ldots H_k \vdash Q
\]

if and only if
for some propositional expression $P$. This is also known as proof by **contradiction** or **reductio ad absurdum**.

- **Contradiction** $[(P \land \sim Q) \rightarrow \sim P \lor (P \land \sim Q) \rightarrow Q]$ A proof by contradiction is based on the tautology $P \leftrightarrow (\sim P \rightarrow (Q \land \sim Q))$. Here the hypotheses $P$ to be true but the conclusion $Q$ false, and from this reach some type of contradiction, either contradicting the assumption $P$ or contradicting the denial $\sim Q$.

**Example 6**: Prove $(P \lor Q) \rightarrow R$, $(R \lor S) \rightarrow \sim P \vdash \sim P$

Proof:

1. $(P \lor Q) \rightarrow R$ Hyp
2. $(R \lor S) \rightarrow \sim P$ Hyp
   \hspace{1cm} < Goal: $\sim P$>
3. $\sim \sim P$ Assumption
4. $P$ 3 DN
5. $P \lor Q$ 4 ADD
6. $R$ 5, 1 MP
7. $R \lor S$ 6 ADD
8. $\sim P$ 7, 2 MP
9. $\sim P \land P$ 4, 9 Conj
10. $\sim P$ 3-9 IDM

END.

- **Reductio ad absurdum** $[(P \land \sim Q) \rightarrow R \land \sim R]$.  

**Reductio ad absurdum** is another form of proof by contradiction. After assuming $P$ is true and $Q$ is not true, one seeks to prove an absurd\textsuperscript{4} result (like $1 = 0$ or $x^2 < 0$), which we denote by $R \land \sim R$. 

\[
H_1, H_2, \ldots, H_k, \sim Q \vdash P \land \sim P
\]
In Example 2 we proved that the square of an odd natural number is odd. We now show the converse is also true; if $n^2$ is odd so is $n$.

Example 7 (Proof by Contrapositive) :

Let $n$ be a natural number. If $n^2$ is odd so is $n$.

Proof:

The proof of this theorem is best carried out by contrapositive which means we assume $n$ is even and show $n^2$ is even. But if $n$ is even it can be written as $n = 2k$ for some natural number $k$. And squaring gives $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which shows that $n^2$ is also even. Hence $(n^2$ odd $) \rightarrow (n$ odd $)$.

END

Margin Note: Not Mathematical Proofs:

- The proof is so easy we’ll skip it.
- Don’t be stupid, of course it’s true!
- It’s true because I said it’s true!
- Oh God let it be true!
- I have this gut feeling.
- I did it last night.
- I define it to be true!

You can think of a few yourself.

Example 7 (Proof by Contrapositive) :

Let $n$ be a natural number. If $n^2$ is odd so is $n$.

Proof:

The proof of this theorem is best carried out by contrapositive which means we assume $n$ is even and show $n^2$ is even. But if $n$ is even it can be written as $n = 2k$ for some natural number $k$. And squaring gives $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which shows that $n^2$ is also even. Hence $(n^2$ odd $) \rightarrow (n$ odd $)$.

END

Margin Note: There are theorems and then there are Theorems. In the Classification Theorem for Modular Simple Lie Algebras the proof required the work of hundreds of mathematicians and consists of an aggregate of several hundreds of papers. If the theorem were to be written out it is estimated it would take between 5000 and 10000 pages.
Example 8: (Proof by Contradiction): \( \sqrt{2} \) is Irrational number.

Proof:

Assume the contrary that \( \sqrt{2} \) is rational number. That means that \( \sqrt{2} = \frac{p}{q} \) where \( p \) and \( q \) are relatively prime integers (that also means that the fraction is reduced to lowest form). If we square both sides of the equation, gives

\[(\sqrt{2})^2 = \left(\frac{p}{q}\right)^2 \text{ which implies } 2 = \frac{p^2}{q^2}\]

Multiplying both sides of this equation by \( q^2 \) yields \( 2q^2 = p^2 \). The last equation tells us that \( p^2 \) is even integer and by the theorem (example 4) \( p \) is even too. We can write \( p = 2k \). After substitution one obtains \( 2q^2 = (2k)^2 \) or \( 2q^2 = 4p^2 \). We can reduce both side of last equation by \( 2 \) which results \( q^2 = 2p^2 \) and one more time we apply the theorem from example 4 leads that \( q \) is even too. But wait! This can’t be since if both \( p \) and \( q \) are even then \( \frac{p}{q} \) is not in the reduced terms.

Therefore, our assumption that \( \sqrt{2} \) is rational is faulty and so the contrary statement is true: the square root of 2 is irrational number. END

Margin Note: Many of the most important theorems in mathematics are proven by contradiction. There of the most famous are:

▶ Cantor’s seminal theorem that the real numbers are uncountable.
▶ Euclid’s proof that there are an infinite number of prime numbers.
▶ Pythagorean’s proof that 2 is irrational.

Example 9: (Proof by Contradiction)

There are an infinite number of prime numbers.

Proof:

Assume the contrary; that there are only a finite number of prime numbers and let \( k \) be the largest prime number. Thus, the prime numbers can be
enumerated 2, 3, 5, 7, 11, 13, \ldots, k. Now consider the product \( P = 235 \ldots k \). Clearly \( P \) is divisible by each of the prime numbers, 2, 3, \ldots, k. But \( P + 1 \) is not divisible by any of the prime numbers 2, 3, \ldots, k since it would have a remainder of 1 if divided by any prime number. But since \( P + 1 \) cannot be divided by any prime number, it must itself be prime. But \( P + 1 \) is larger than k which contradicts the assumption that k is the largest prime number. Hence there are an infinite number of prime numbers. \hspace{1cm} \textbf{END}

\textbf{Example 10: (Proof by Contradiction):} Prove that if a, b, and c are odd integers, then the quadratic equation

\[ ax^2 + bx + c = 0. \]

does not have a rational solution.

\textbf{Proof:}

We assume a, b, c are odd integers and that p, q \( \neq 0 \) are integers such that \( x = p/q \) is a solution of the quadratic with no common factors greater than 1. Hence, we have

\[ a(p/q)^2 + b(p/q) + c = 0. \]

or \[ ap^2 + bpq + cq^2 = 0. \] The idea is to show that both p, q are odd and hence \( ap^2, bpq, cq^2 \) will be odd and so their sum cannot be zero, which contradicts the fact that \( p/q \) is a solution. To prove p, q are odd, first assume p is even. But that means \( ap^2 + bpq \) is even which means \( cq^2 \) is even which cannot be since c and q are both odd (q must be odd else \( p/q \) will have a common factor). Therefore p is odd. A similar argument will show that q is odd as well (this sentence is very frequently used in mathematical literature as a part of proof, even though most of faulty proofs are hidden here). \hspace{1cm} \textbf{END}

\textbf{Example 11: (Reductio ad absurdum):}

For any real numbers a, b, if \( a = b \), then \( a^2 = b^2 \)

\textbf{Proof:}

Assume \( a = b \) and \( a^2 \neq b^2 \). From these assumptions we have

\[ a^2 - b^2 = (a - b)(a + b) \neq 0. \]
But this inequality implies \( a - b \neq 0 \) or \( a \neq b \) which contradicts the assumption \( a = b \). Hence, we conclude that \( a^2 = b^2 \).

**Tips for Proving Theorems:** Here are a few guidelines that might be useful for proving theorems.

- Draw figures to visualize the concepts.
- Construct examples that illustrate the general principles.
- Work backwards (i.e. what are the steps before the conclusion).
- For " \( \rightarrow \) " theorems ask if the converse " \( \leftarrow \) " is true.
- Modify the theorem to make it easier.
- Generalize i.e. does the theorem hold in more general situations?

If a theorem is stated in two dimensions, does it hold in three dimensions?
- Did you actually use all the assumptions?

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**Necessary and Sufficient Conditions**

Many theorems in mathematics are stated in the form \( P \leftrightarrow Q \), which means " \( P \) if and only if \( Q \)" or often stated that " \( P \) is a necessary and sufficient condition for \( Q \)." A theorem of this type means two things: \( P \rightarrow Q \) and \( Q \rightarrow P \) so to prove a theorem of this type you must prove two implications. The following theorem illustrates this idea.

**Example 12:** (If and Only If) Let \( n \) be any integer, then 3 divides \( 2n - 1 \) if and only if 3 does not divide \( n \).

**Proof:**

We prove two implications.

\[
(3 \text{ divides } n^2 - 1) \rightarrow (3 \text{ does not divide } n): 
\]

Since 3 is a prime number and divides \( n^2 - 1 = (n - 1)(n + 1) \) it must divide either
n −1 or n +1. If 3 divides n −1, it cannot divide n (it will have a remainder of 1), and if 3 divides n +1 it cannot divide n (it will have a remainder of 2). Hence 3 does not divide n.

\[(3 \text{ does not divide } n) \Rightarrow (3 \text{ divides } n^2 -1):\]

If 3 does not divide n, then we can write
\[\frac{n}{3} = q + \frac{r}{3}\]
or \[n = 3q + r\], where the remainder \(r\) is either 1 or 2.

If \(r = 1\) then \(n - 1 = 3q\) which means 3 divides \((n - 1)(n - 1) = n^2 + 1\).

If \(r = 2\) then \(n - 2 = 3q\) or equivalently \(n - 1 = 3r + 3 = 3(r + 1)\), which also means 3 divides \(n^2 - 1\).  

Lemmas and Corollaries: In addition to theorems there are \textit{lemmas} and \textit{corollaries}. Although theorems, lemmas and corollaries are the same from a logical point of view, it is how they are used and their importance that distinguishes them.

- A \textit{lemma} is a statement that is proven as an aid in proving a theorem. Often unimportant details are put in a lemma so not to clutter a theorem.
- A \textit{corollary} is a statement that is easily deduced from a theorem.

Often an “if and only if” theorem can be proven without dividing it into two parts. The following example is such a theorem.

\textbf{Example 13.} If \(n\) is a positive integer, then 3 divides \(n\) if and only if 3 divides the sum of the digits of \(n\).

\textbf{Proof:}

We can represent \(n\) as
\[n = a_0 + a_110 + a_210^2 + \ldots + a_{k-1}10^{k-1} + a_k10^k = \]
\[(a_0 + a_1 9 + a_2 99 + \ldots + a_k(10^k - 1)) + (a_0 + a_1 + a_2 + \ldots + a_k - 1 + a_k)\]

It is clear now that 3 divides \(n - s\), where \(s = (a_0 + a_1 + a_2 + \ldots + a_k - 1 + a_k)\) and so if 3 divides \(n\) if and only if 3 divides \(s\). END

Counterexample

At one time there was a “theorem” that stated \(2^n + 1\) is a prime number for natural numbers \(n = 2^k\) until someone made the observation that
\[2^{32} + 1 = 4294967297 = 641 \times 6700417\]
(this is a counterexample for that early “theorem”).
Problems

1. (Direct Proof) Prove the following by a direct proof.

   a) The sum of two even integers is even.

   b) The sum of an even and odd integer is odd.

   c) If \( a \) divides \( b \), and \( b \) divides \( c \), then \( a \) divides \( c \).

   d) The product of two consecutive natural numbers plus the larger number is a perfect square. That is, the square of another natural number.

   e) Every odd integer is the difference between two perfect squares (the square of an integer).

   f) If \( a, b \) are real numbers, then \( a^2 + b^2 \geq 2ab \).

   g) The sum of two rational numbers is rational.

   h) Let \( p(x) \) be a polynomial and \( A \) is the sum of the coefficients of the even powers, and \( B \) the sum of the coefficients of the odd powers. Show \( A + B = p(1) \) and \( A - B = p(-1) \).

2. Prove the following using the Direct Proof Method:

   a) \((P \lor Q) \rightarrow R \models P \rightarrow Q\)

   b) \((\neg P \lor Q) \rightarrow R, \ (R \lor T) \rightarrow S \models \neg P \rightarrow S\)
c) \( P \rightarrow (Q \rightarrow R), \ R \rightarrow (S \land T), \ \vdash \ P \rightarrow (Q \rightarrow T) \)

d) \( P \rightarrow Q, \ R \rightarrow Q \ \vdash \ (P \lor R) \rightarrow Q \)

3. **(Proof by Contradiction)** Prove the following by contradiction

a) If \( n \) is an integer and \( 5n + 2 \) is an even integer, then \( n \) is even.

b) If \( x \) and \( y \) are integers and \( x + y \) is even, then \( x \) and \( y \) have the same parity (i.e. both are even or both are odd).

c) Let \( x \) and \( y \) be integers. If \( xy \) is even, then at least one of \( x \) and \( y \) must be even.

d) Prove that 1 is equal to 0.9999 \( \ldots \).

4. Prove the following using indirect proof methods:

a) \( P \rightarrow Q \ \vdash \ (P \land R) \rightarrow Q \)

b) \( P \rightarrow \lnot (Q \rightarrow \lnot R) \ \vdash \ P \rightarrow R \)

c) \( (P \lor Q) \rightarrow \lnot R, \ S \rightarrow R \ \vdash \ P \rightarrow \lnot S \)

d) \( P \rightarrow Q, \ P \lor (R \rightarrow S), \ \lnot Q \ \vdash \ R \rightarrow S \)

5. Prove the following theorems for integers \( m, n \).

a) 5 divides \( n^4 - 1 \) if and only if 5 does not divide \( n \).

b) 9 divides \( n \) if and only if 9 divides the sum of the digits of \( n \).
c) The product $mn$ is even if and only if at least one of $m$ and $n$ is even.

6. (Counterexamples) A counterexample is an exception to a rule. In mathematics, they are used to probe the boundaries of theorem. A counterexample to a given claim may show that the assumptions are false or incomplete, and thus allow the mathematician to add or adjust the conjectures. Find counterexamples for the following faulty theorems and tell how you could add new hypothesis to make the claim valid.

a) If $a > b$ then $|a| > |b|$. 

b) If $(a - b)^2 = (m - n)^2$ then $a - b = m - n$.

c) If $x$ and $y$ are real numbers, then $\sqrt{xy} = \sqrt{x} \cdot \sqrt{y}$.

d) If $f$ is a continuous real-valued function defined on $[a, b]$, then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

7. (Euler Diagrams) The Swiss mathematician **Leonard Euler** (1707-1783) used diagrams as an aid to understanding mathematical arguments, which consists of drawing circles, denoted $P$, $Q$ and so on to represent situations when sentences $P$ and $Q$ were true. For example, the sentence $P \rightarrow Q$ would be illustrated by a set $P$ being a subset of $Q$ as in “if it’s an apple, then it’s a fruit.”
Contrary statements which cannot both be true for the same object, such as

$$P : \text{It is a cat}$$

$$Q : \text{It is a dog}$$

would be represented by disjoint sets. Logical negation of sentences would be drawn as set complements in Euler’s diagrams.