5.5 Dual problem: minimization with problem constraints of the form $\geq$.

The procedures for the simplex method will be illustrated through an example. Be sure to read the textbook to fully understand all the concepts involved.
Transpose and Main Properties

1. Given a matrix $A$. The **transpose** of $A$, denoted $A^T$, is the matrix formed by interchanging the rows and corresponding columns of $A$ (first row with first column, second row with second column, and so on.)

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(kA)^T = k(A^T)$
5.5 Dual problem: minimization with problem constraints of the form $\geq$

Associated with each minimization problem with constraints is a maximization problem called the **dual problem**. The dual problem will be illustrated through an example. Read the textbook carefully to learn the details of this method. We wish to minimize the objective function subject to certain constraints:

\[
C = 16x_1 + 9x_2 + 21x_3 \\
x_1 + x_2 + 3x_3 \geq 12 \\
2x_1 + x_2 + x_3 \geq 16 \\
x_1, x_2, x_3 \geq 0
\]
FORMATION OF THE DUAL PROBLEM

Given a minimization problem with $\geq$ problem constraints:

**Step 1.** Use the coefficients and constants in the problem constraints and the objective function to form a matrix $A$ with the coefficients of the objective function in the last row.

**Step 2.** Interchange the rows and columns of matrix $A$ to for the matrix $A^T$, the transpose of $A$.

**Step 3.** Use the rows of $A^T$ to form a maximization problem with $\leq$ problem constraints.
### Initial matrix

We start with an initial matrix $A$, corresponds to the problem constraints:

$$
A = \begin{bmatrix}
1 & 1 & 3 & 12 \\
2 & 1 & 1 & 16 \\
16 & 9 & 21 & 1
\end{bmatrix}
$$
Transpose of matrix A

To find the transpose of matrix A, interchange the rows and columns so that the first row of A is now the first column of A transpose.

\[
A^T = \begin{bmatrix}
1 & 2 & 16 \\
1 & 1 & 9 \\
3 & 1 & 21 \\
12 & 16 & 1 \\
\end{bmatrix}
\]
Dual of the minimization problem is the following maximization problem:

Maximize \( P \) under the following constraints.

\[
P = 12 y_1 + 16 y_2 \\
y_1 + 2 y_2 \leq 16 \\
y_1 + y_2 \leq 9 \\
3 y_1 + y_2 \leq 21 \\
y_1, y_2 \geq 0
\]
THE FUNDAMENTAL PRINCIPLE OF DUALITY

Theorem: A minimization problem has a solution if and only if its dual problem has a solution. If a solution exists, then the optimal value of the minimization problem is the same as the optimal value of the dual problem.
**SOLUTION OF A MINIMIZATION PROBLEM**

1. Write all problem constraints as \( \geq \) inequalities. (This may introduce negative numbers on the right side.)

2. Form the dual problem.

3. Write the initial system of the dual problem, using the variables from the minimization problem as the slack variables.

4. Use the simplex method to solve the dual problem.

5. Read the solution of the minimization problem from the bottom row of the final simplex tableau in Step (iv).

   [Note: If the dual problem has no solution, then the minimization problem has no solution.]
Forming the Dual problem with slack variables $x_1, x_2, x_3$

result:

\[
\begin{align*}
y_1 + 2y_2 + x_1 &= 16 \\
y_1 + y_2 + x_2 &= 9 \\
3y_1 + y_2 + x_3 &= 21 \\
-12y_1 - 16y_2 + p &= 0
\end{align*}
\]
Form the simplex tableau for the dual problem and determine the pivot element

The first pivot element is 2 (in red) because it is located in the column with the Smallest negative number at the bottom(-16) and when divided into the rightmost constants, yields the smallest quotient (16 divided by 2 is 8)
Divide row 1 by the pivot element (2) and change the exiting variable to $y_2$ (in red)

Result:

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2$</td>
<td>.5</td>
<td>1</td>
<td>.5</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>$x_3$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td>$P$</td>
<td>-12</td>
<td>-16</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Pivot Operations (Pivoting)

- $-1 \times R_1 + R_2 \rightarrow R_2$
- $-1 \times R_1 + R_3 \rightarrow R_3$
- $16 \times R_1 + R_4 \rightarrow R_4$
Perform row operations to get zeros in the column containing the pivot element. Identify the next pivot element (0.5) (in red)

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_2$</td>
<td>.5</td>
<td>1</td>
<td>.5</td>
<td>0</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>$x_2$</td>
<td>.5</td>
<td>0</td>
<td>-.5</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>2.5</td>
<td>0</td>
<td>-.5</td>
<td>0</td>
<td>1</td>
<td>13</td>
</tr>
<tr>
<td>$P$</td>
<td>-4</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>128</td>
</tr>
</tbody>
</table>
Variable $y_1$ becomes new entering variable

Divide row 2 by 0.5 to obtain a 1 in the pivot position.

\[
\begin{bmatrix}
y_2 & 0.5 & 1 & 0.5 & 0 & 0 & 8 \\
y_3 & 1 & 0 & -1 & 2 & 0 & 2 \\
x_3 & 2.5 & 0 & -0.5 & 0 & 1 & 13 \\
P & -4 & 0 & 8 & 0 & 0 & 128 \\
\end{bmatrix}
\]

- $0.5R_2 + R_1 \rightarrow R_1$
- $-2.5R_2 + R_3 \rightarrow R_3$
- $4R_2 + R_4 \rightarrow R_4$
More row operations

-0.5*R₂ + R₁ → R₁
-2.5*R₂ + R₃ → R₃
4*R₂ + R₄ → R₄

\[
\begin{bmatrix}
 y_1' & y_2 & x_1 & x_2 & x_3 & P \\
 0 & 1 & 1 & -1 & 0 & 8 \\
 1 & 0 & -1 & 2 & 0 & 2 \\
 0 & 0 & 2 & -5 & 1 & 8 \\
 0 & 0 & 4 & 8 & 0 & 136 \\
\end{bmatrix}
\]
Solution: An optimal solution to a minimization problem can always be obtained from the bottom row of the final simplex tableau for the dual problem.

Minimum of $P$ is 136. It occurs at $x_1 = 4$, $x_2 = 8$, $x_3 = 0$.
Example (Problem#23).

The matrices corresponding to the given problem and the dual problem are:

\[
A = \begin{bmatrix}
1 & 1 & 4 \\
1 & -2 & -8 \\
-2 & 1 & -8 \\
7 & 5 & 1
\end{bmatrix}
\]
\[
A^T = \begin{bmatrix}
1 & 1 & -2 & 7 \\
1 & -2 & 1 & 5 \\
4 & -8 & -8 & 1
\end{bmatrix}
\]

Thus, the dual problem is:
Maximize \( P = 4y_1 - 8y_2 - 8y_3 \)
Subject to:
\[
y_1 + y_2 - 2y_3 \leq 7
\]
\[
y_1 - 2y_2 + y_3 \leq 5
\]
\[
y_1, y_2, y_3 \geq 0
\]
We introduce slack variables $x_1$ and $x_2$ to obtain the initial system:

$$y_1 + y_2 - 2y_3 + x_1 = 7$$
$$y_1 - 2y_2 + y_3 + x_2 = 5$$
$$-4y_1 + 8y_2 + 8y_3 + P = 0$$

The simplex tableau for this problem is:

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>3</td>
<td>-3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$P$</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>3</td>
<td>-3</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$P$</td>
<td>0</td>
<td>0</td>
<td>12</td>
<td>0</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

$P = 0$
1) \( R_2 + R_1 \rightarrow R_1 \) and \( 4R_2 + R_3 \rightarrow R_3 \)

Optimal solution: \( \min C = 20 \) at \( x_1 = 0, x_2 = 4 \).
**SIMPLEX ALGORITHM FOR STANDARD MAXIMIZATION PROBLEMS**

**Step 1:**
Write the standard maximization problem in standard form, introduce slack variables to form the initial system, and write the initial tableau.

**Step 2:**
Are there any negative indicators in the bottom row?

**Step 3:**
Select the pivot column.

**Step 4:**
Are there any positive elements in the pivot column above the dashed line?

**Step 5:**
Select the pivot element and perform the pivot operation.

Stop
The optimal solution has been found.

Stop
The linear programming problem has no optimal solution.