In this section, we will explore applications which utilize the graph of a system of linear inequalities.
A familiar example

We have seen this problem before. An extra condition will be added to make the example more interesting. Suppose a manufacturer makes two types of skis: a trick ski and a slalom ski. Suppose each trick ski requires 8 hours of design work and 4 hours of finishing. Each slalom ski 8 hours of design and 12 hours of finishing. Furthermore, the total number of hours allocated for design work is 160 and the total available hours for finishing work is 180 hours. Finally, the number of trick skis produced must be less than or equal to 15. How many trick skis and how many slalom skis can be made under these conditions?

Now, here is the twist: Suppose the profit on each trick ski is $5 and the profit for each slalom ski is $10. How many each of each type of ski should the manufacturer produce to earn the greatest profit?
Linear Programming problem

This is an example of a **linear programming problem**. Every linear programming problem has two components:

1. **A linear objective function is to be maximized or minimized.** In our case the objective function is
   \[ \text{Profit} = 5x + 10y \] (5 dollars profit for each trick ski manufactured and $10 for every slalom ski produced).

2. **A collection of linear inequalities that must be satisfied simultaneously.** These are called the **constraints** of the problem because these inequalities give limitations on the values of \( x \) and \( y \). In our case, the linear inequalities are the constraints.

   \[
   \begin{align*}
   0 &\leq x \leq 15 \\
   x &\geq 0 \\
   y &\geq 0 \\
   8x + 8y &\leq 160 \\
   4x + 12y &\leq 180
   \end{align*}
   \]

   - \( x \) and \( y \) have to be positive
   - The number of trick skis must be less than or equal to 15
   - Design constraint: 8 hours to design each trick ski and 8 hours to design each slalom ski. Total design hours must be less than or equal to 160
   - Finishing constraint: Four hours for each trick ski and 12 hours for each slalom ski.
One square = 5 units.

\[ x \geq 0 \quad \text{These first two inequalities include only points in the first quadrant, including the } x \text{ and } y \text{ axes} \]
\[ y \geq 0 \]
\[ x \leq 15 \]
\[ 8x + 8y \leq 160 \]
\[ 4x + 12y \leq 180 \]

The origin satisfies all the inequalities. We shade the region to the left of the vertical line \( x = 15 \) and below the other lines. The intersection of all graphs is the yellow shaded region.

Some possible solutions include the points (1,1), (2,1), (1,2) . (1,1) means produce 1 of each type of ski. (2,1) means 1 trick ski and 1 slalom ski. (1,2) means one trick ski and 2 slalom skis produced.
Linear programming

3. The **feasible set** is the set of all points that are possible for the solution. In this case, we want to determine the value(s) of \( x \), the number of trick skis and \( y \), the number of slalom skis that will yield the maximum profit. Only certain points are eligible. Those are the points within the common region of intersection of the graphs of the constraining inequalities. Let’s return to the graph of the system of linear inequalities. Notice that the feasible set is the yellow shaded region.

Our task is to maximize the profit function

\[
P = 5x + 10y\]

by producing \( x \) trick skis and \( y \) slalom skis, but use only values of \( x \) and \( y \) that are within the yellow region graphed in the next slide.
Profit = 5x + 10y

Scale: One square = 5 units.

\[ x \geq 0 \quad \text{These first two inequalities include only points in the first quadrant, including the x and y axes} \]
\[ y \geq 0 \]
\[ x \leq 15 \]
\[ 8x + 8y \leq 160 \]
\[ 4x + 12y \leq 180 \]

The origin satisfies all the inequalities. We shade the region to the left of the vertical line \( x = 15 \) and below the other lines. The intersection of all graphs is the yellow shaded region.

Some possible solutions include the points (1,1), (2,1), (1,2), (1,1) means produce 1 of each type of ski, (2,1) means 2 trick skis and 1 slalom ski, (1,2) means one trick ski and 2 slalom skis produced.

Feasible set
Maximizing the profit

Profit = 5x + 10y Suppose profit equals a constant value, say k. Then the equation k = 5x + 10y represents a family of parallel lines each with slope of one-half. For each value of k (a given profit), there is a unique line. What we are attempting to do is to find the largest value of k possible. The graph on the next slide shows a few iso-profit lines. Every point on this profit line represents a production schedule of x and y that gives a constant profit of k dollars. As the profit k increases, the line shifts upward by the amount of increase while remaining parallel. The maximum value of profit occurs at what is called a corner point - a point of intersection of two lines. The exact point of intersection of the two lines is (7.5, 12.5). Since x and y must be whole numbers, we round the answer down to (7,12). See the graph in the next slide.
Profit = 5x + 10y

x \geq 0
\ y \geq 0
\ x \leq 15
8x + 8y \leq 160
4x + 12y \leq 180

Scale: One square = 5 units.

These first two inequalities include only points in the first quadrant, including the x and y axes.

The origin satisfies all the inequalities. We shade the region to the left of the vertical line x = 15 and below the other lines. The intersection of all graphs is the yellow shaded region.

Various profit levels are indicated by the pink lines. Each line has a slope of -1/2 the slope of the line 5x + 10y = P, where P is a constant. The profit level that is greatest and yet still includes one or more points of the feasible set is indicated.

Some possible solutions include the points (1,1), (2,1), (1,2). (1,1) means produce 10 each type of ski. (2,1) means: 2 trick skis and 1 slalom ski. (1,2) means one trick ski and 2 slalom skis produced.
Maximizing the Profit

Thus, the manufacturer should produce 7 trick skis and 12 slalom skis to achieve maximum profit. What is the maximum profit?

\[ P = 5x + 10y \]

\[ P = 5(7) + 10(12) = 35 + 120 = 155 \]
General Result

If a linear programming problem has a solution, it is located at a vertex of the set of feasible solutions. If a linear programming problem has more than one solution, at least one of them is located at a vertex of the set of feasible solutions.

If the set of feasible solutions is bounded, as in our example, then it can be enclosed within a circle of a given radius. In these cases, the solutions of the linear programming problems will be unique.

If the set of feasible solutions is not bounded, then the solution may or may not exist. Use the graph to determine whether a solution exists or not.
General Procedure for Solving Linear Programming Problems

1. Write an expression for the quantity that is to be maximized or minimized. This quantity is called the **objective function** and will be of the form \( z = Ax + By \). In our case \( z = 5x + 10y \).

2. Determine all the constraints and graph them.

3. Determine the feasible set of solutions— the set of points which satisfy all the constraints simultaneously.

4. Determine the vertices of the feasible set. Each vertex will correspond to the point of intersection of two linear equations. So, to determine all the vertices, find these points of intersection.

5. Determine the value of the objective function at each vertex.
Linear programming problem with no solution

Maximize the quantity \( z = x + 2y \) subject to the constraints \( x + y \geq 1, \ x \geq 0, \ y \geq 0 \)

1. The objective function is \( z = x + 2y \) is to be maximized.

2. Graph the constraints: (see next slide)

3. Determine the feasible set (see next slide)

4. Determine the vertices of the feasible set. There are two vertices from our graph: \((1,0)\) and \((0,1)\)

5. Determine the value of the objective function at each vertex.

6. at \((1,0)\): \( z = (1) + 2(0) = 1 \)
   
   at \((0,1)\): \( z = 0 + 2(1) = 2 \).

We can see from the graph there is no feasible point that makes \( z \) largest. We conclude that the linear programming problem has no solution.
The set of feasible solutions is unbounded. The region has two vertices (0, 1) and (1, 0).

\[ x + 2y = z = 4 \]
\[ x + 2y = z = 3 \]

There is no feasible point within the region that will make \( z = x + 2y \) the largest. The objective function has no maximum value but does have a minimum value.

Scale: two squares equals one unit.
LINEAR PROGRAMMING PROBLEM

is a problem concerned with finding the maximum or minimum value of a linear OBJECTIVE FUNCTION of the form

\[ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n, \]

where the DECISION VARIABLES \( x_1, x_2, \ldots, x_n \) are subject to PROBLEM CONSTRAINTS in the form of linear inequalities and equations. In addition, the decision variables must satisfy the NONNEGATIVE CONSTRAINTS

\[ x_i \geq 0, \text{ for } i = 1, 2, \ldots, n. \]

The set of points satisfying both the problem constraints and the nonnegative constraints is called the FEASIBLE REGION for the problem. Any point in the feasible region that produces the optimal value of the objective function over the feasible region is called an OPTIMAL SOLUTION.
CONSTRUCTING THE MATHEMATICAL MODEL

For the applied Linear programming problem

1. Introduce decision variables

2. Summarize relevant material in table form, relating the decision variables with the columns in the table, if possible.

3. Determine the objective and write a linear objective function.

4. Write problem constraints using linear equations and/or inequalities.

5. Write non-negative constraints.
FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING

If the optimal value of the objective function in a linear programming problem exists, then that value must occur at one (or more) of the corner points of the feasible region.
EXISTENCE OF SOLUTIONS

(A) If the feasible region for a linear programming problem is bounded, then both the maximum value and the minimum value of the objective function always exist.

(B) If the feasible region is unbounded, and the coefficients of the objective function are positive, then the minimum value of the objective function exists, but the maximum value does not.

(C) If the feasible region is empty (that is, there are no points that satisfy all the constraints), the both the maximum value and the minimum value of the objective function do not exist.
GEOMETRIC SOLUTION OF A LINEAR PROGRAMMING PROBLEM WITH TWO DECISION VARIABLES

Step 1. Graph the feasible region. Then, if according to 4 an optimal solution exists, find the coordinates of each corner point.

Step 2. Make a table listing the value of the objective function at each corner point.

Step 3. Determine the optimal solution(s) from the table in Step (2).

Step 4. For an applied problem, interpret the optimal solution(s) in terms of the original problem.
**Example (33)**

The graphs of the inequalities are shown below. The solution is indicated by the shaded region. The solution region is *bounded*.

The corner points of the solution region are:
- (0, 0), the intersection of $x = 0, y = 0$;
- (0, 6), the intersection of $x = 0, x + 2y = 12$;
- (2, 5), the intersection of $x + 2y = 12, x + y = 7$;
- (3, 4), the intersection of $x + y = 7, 2x + y = 10$;
- (5, 0), the intersection of $y = 0, 2x + y = 10$. 

Graph of example (33)

2 \(x + y = 10\)

\(x + 2y = 12\)

\(x + y = 7\)

Bounded
Example (Problem #21 Chapter 5.2)

**Step (1):** Graph the feasible region and find the corner points. The feasible region $S$ is the solution set of the given inequalities, and is indicated by the shading in the graph at the right. The corner points are $(3, 8), (8, 10),$ and $(12, 2)$. Since $S$ is bounded, it follows that $P$ has a maximum value and a minimum value.
Bounded

-2 \( x_1 \) + 5 \( x_2 \) = 34

2 \( x_1 \) + \( x_2 \) = 26

2 \( x_1 \) + 3 \( x_2 \) = 30

\((3, 8)\)

\((8, 10)\)

\((12, 2)\)
Example (continue)

**Step (2):** Evaluate the objective function at each corner point. The value of \( P \) at each corner point is

\[
\begin{align*}
(3, 8) & : P = 20(3) + 10(8) = 140 \\
(8, 10) & : P = 20(8) + 10(10) = 260 \\
(12, 2) & : P = 20(12) + 10(2) = 260
\end{align*}
\]

**Step (3):** Determine the optimal solutions.

The minimum occurs at \( x_1 = 3, x_2 = 8 \), and the minimum value is \( P = 140 \);

the maximum occurs at \( x_1 = 8, x_2 = 10 \), at \( x_1 = 12, x_2 = 2 \), and at any point along the line segment joining \( (8, 10) \) and \( (12, 2) \). The maximum value is \( P = 260 \).